Measure Theory with Ergodic Horizons
Lecture 10

Before proving the tightness theorem, let's recall equivalent conditions to compactness in metric spaces.

Theorem. For any metric space (X,d), the following are equivalent:

(1) X is compact, i.e. every open cover has a finite subcover.
(2) X is sequentially compact, i.e. every sequence has a convergent subsequence.
(3) X is complete and **totally bounded**, i.e. for each $\varepsilon>0$ there is a finite **ε-net**, i.e. a cover of X consisting of balls of radius ε.

Corollary. Thus, in a complete metric space, compact subsets are exactly the closed and totally bounded ones.

Theorem. Every finite Borel measure μ on a Polish metric space X is tight.

Proof. Let B be a measurable set. By strong regularity, there is a closed set $C\subseteq B$ with $\mu(B\setminus C)<\varepsilon/2$, so it's enough to find a compact subset $K\subseteq C$ with $\mu(C\setminus K)<\varepsilon/2$. In other words, we may assume that B is closed. Since B is still Polish with the same metric as X, we may assume $B=X$. Fix $\varepsilon>0$.

Let (ε_n) be a positive sequence converging to 0, e.g. $(\frac{1}{n})$. For each $n\in\mathbb{N}$, let $(B_k^{\varepsilon_n})$ be a cover of X by closed balls $B_k^{\varepsilon_n}$ of radius ε_n (such a sequence exists by the separability of X). Note that $\bigcup_{k<K} B_k^{\varepsilon_n}$ has measure $\approx_{\varepsilon/2^{n+1}} \mu(X)$ for all large enough K, by monotone convergence and because $\mu(X)<\infty$. Let $C_n := \bigcup_{k<K_n} B_k^{\varepsilon_n}$ where K_n is large enough. Finally, take $K=\bigcap_{n\in\mathbb{N}} C_n$.

Then K is closed being an intersection of closed sets, and it is totally bounded by construction: for each ε_n, $\{B_k^{\varepsilon_n}\}_{K_n}$ is a finite ε_n-net for C_n, hence also for K. Lastly, $\mu(X\setminus K)=\mu\left(\bigcup_n (X\setminus C_n)\right)\leq\sum_n \mu(X\setminus C_n)=\sum_n \varepsilon\cdot 2^{-(n+1)}=\varepsilon$. $\square$

Corollary (Strong reg. and tightness for some σ-finite measures). Let X be a Polish space. Any locally finite Borel measure μ on X is strongly regular and tight.

Proof. X is 2^{nd} ctbl, so locally finite $\Rightarrow$ σ-finite by open sets, hence μ is strongly regular. Recall/learn that open subsets of Polish are Polish but with a different equivalent metric, hence X is σ-finite by Polish subsets, using which one can deduce tightness, just like in the proof of strong regularity for σ-finite by open sets. HW. $\square$

99% lemmas.

Percentage pigeonhole principle. Let $(X, \mathcal{B}, \mu)$ be a measure space and let M be a finite measure set. Suppose that $M \subseteq \bigsqcup_{n \in \mathbb{N}} B_n$ where each B_n is measurable and such that $\mu(M) \geq 0.99\, \mu\left(\bigsqcup_{n \in \mathbb{N}} B_n\right)$, i.e. M is $\geq 99\%$ of $\bigsqcup_{n \in \mathbb{N}} B_n$. Then there is B_n whose $\geq 99\%$ is M, i.e.

$$\frac{\mu(M \cap B_n)}{\mu(B_n)} \geq 0.99.$$

Proof. Pots of soup and percentage of carrots. $\square$

99% lemmas.

(a) Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $\mathcal{A} \subseteq \mathcal{B}$ be an algebra generating $\mathcal{B}$. Then for each positive measure set $M \subseteq X$ there is $A \in \mathcal{A}$ of finite positive measure with $\mu(M \cap A) \geq 0.99\, \mu(A)$.

(b) For $\mathbb{R}^d$ with Lebesgue measure λ and for every positive measure set $M \subseteq \mathbb{R}^d$ there are arbitrarily small boxes B of finite measure such that

$$\mu(M \cap B) \geq 0.99\, \mu(B).$$

(c) Let μ_p denote the Bernoulli(p) measure on $2^{\mathbb{N}}$. For every positive measure set $M \subseteq 2^{\mathbb{N}}$ there a arbitrarily small (in measure) cylinders C such that

$$\mu(M \cap C) \geq 0.99\,\mu(C).$$

Proof. (a) By the uniqueness in Carathéodory's thm, we know that $\mu =$ the outer measure μ^* defined by the algebra $\mathcal{A}$. Fix a positive measure set M and make it smaller to make it have finite positive measure (by σ-finiteness, $X = \bigcup_n B_n$ where $\mu(B_n) < \infty$, and $0 < \mu(M \cap B_n) < \infty$ for some n by ctbl subadditivity). By the def. of μ^*, there is $\{A_n\} \subseteq \mathcal{A}$ such that

$$\mu\Big(\bigcup_{n \in \mathbb{N}} A_n \setminus M\Big) < 0.01\,\mu(M) \leq 0.01 \cdot \mu\Big(\bigcup_{n \in \mathbb{N}} A_n\Big).$$

Disjointifying we may assume that the A_n are disjoint, hence the percentage pigeonhole gives some A_n with $\mu(M \cap A_n) \geq 0.99 \cdot \mu(A_n)$.

(b) Follows from (a) applied to the algebra $\mathcal{A}$ of finite disjoint unions of boxes, noting that each box of finite measure is itself a finite disjoint of boxes of measure $< \varepsilon$, so every $A \in \mathcal{A}$ is a fin. disj. union of boxes of measure $< \varepsilon$.

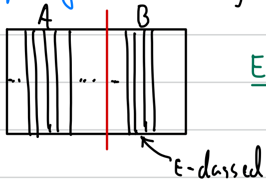
(c) Follows from (a) applied to the algebra of finite unions of cylinders, noting that each cylinder is a finite union of cylinders of measure $< \varepsilon$. $\square$

Remark. The 99% cannot be replaced by 100% (i.e. all except null) because there are closed sets with empty interior and positive measure, e.g. some Cantor sets in $\mathbb{R}$.

Applications to ergodicity.

Def. Let $(X, \mathcal{B}, \mu)$ be a measure space. An equivalence relation E on X is called

μ-ergodic if every E-invariant (i.e. union of E-orbits) measurable set is null or conull.



$\mu(A)\cdot\mu(B)=0$

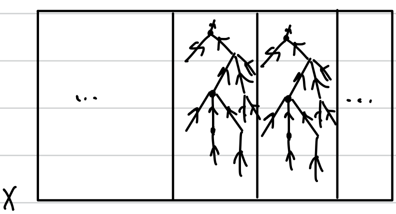
Examples. (a) Most equivalence relations arise as orbit equivalence relations of actions of groups. Let $\Gamma \curvearrowright X$ be an action of a group Γ on a set X. The orbit equivalence relation E_Γ of this action is the relation of being in the same Γ-orbit, i.e. for $x,y \in X$,

$$x \, E_\Gamma \, y \;:\iff\; y = \gamma . x \text{ for some } \gamma \in \Gamma.$$

A single Γ-orbit is $[x]_\Gamma := \Gamma . x := \{\gamma . x : \gamma \in \Gamma\}$, the same as the E_Γ-class of x.

(b) Let $T : \overset{[0,1]}{X} \to \overset{[0,1]}{X}$ be any map. The orbit equivalence relation of T, denoted by E_T, is defined by: $x \, E_T \, y$ $:\iff$ x and y are in the same G_T connected component
$\iff T^n x = T^m y$ for some $n,m \in \mathbb{N}$.

Consider the graph of T as an actual graph G_T with vertex set X and directed edges of the form (x, Tx), $x \in X$. This is an acyclic graph is T is aperiodic, i.e. $T^n x \neq x$ for all $n \geq 1$ and $x \in X$.



If T is an aperiodic bijection, then each connected component is a "$\mathbb{Z}$-line", i.e. a bi-infinite line.

We'll show next time that the Vitally equivalence relation E_V on $\mathbb{R}$ is ergodic wrt Lebesgue measure.